國 立 雲 林 科 技 大 學

111 學年度第二學期工程學院會考試題

時間：10:10~12:00
科目：微積分
班級：

※本次測驗禁止使用電子計算機

1. 求下列積分：(7% × 10 = 70%)

(a) $\int x^2 \ln(x-1)\,dx$

(b) $\int_2^4 3^x\,dx$

(c) $\int (e^x + x)^2\,dx$

(d) $\int \sin^7 x \cos^2 x\,dx$

(e) $\int \frac{1}{x^2\sqrt{9-4x^2}}\,dx$

(f) $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

(g) $\int \frac{x^3-2x-4}{x^4+x^2}\,dx$

(h) $\int_1^\infty \frac{x+1}{x^3}\,dx$

(i) $\int x\cos x\,dx$

(j) $\int_1^{29} \frac{1}{\sqrt[3]{x-2}}\,dx$

2. 求直線 $y = x - 1$ 和拋物線 $y^2 = 2x + 6$ 所包圍的面積。

(a) 對 y 作積分(左右法)。 (5%)

(b) 對 x 作積分(上下法)。 (5%)

3. 求 $y = x$ 和 $y = x^2$ 所包圍的區域繞 $y = 2$ 旋轉所得旋轉體的體積。 (10%)

4. 求拋物線 $y^2 = x$ 從 (0,0) 到 (1,1) 的弧長。

(已知 $\int \sec\theta\,d\theta = \ln|\tan\theta + \sec\theta| + C$) (10%)

《試題完》

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1. Find the integral：(7% × 10 = 70%)

(a) $\int x^2 \ln(x-1)\,dx$

(b) $\int_2^4 3^x\,dx$

(c) $\int (e^x + x)^2\,dx$

(d) $\int \sin^7 x \cos^2 x\,dx$

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(h) $\int_1^\infty \frac{x+1}{x^3}\,dx$

(i) $\int x \cos x\,dx$

(j) $\int_1^{29} \frac{1}{\sqrt[3]{x-2}}\,dx$

2. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

   (a) Integrating over *y* direction. (5%)

   (b) Integrating over *x* direction. (5%)

3. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume of the resulting solid. (10%)

4. Find the length of the arc of the parabola $y^2 = x$ from (0,0) to (1,1). ($\int \sec\theta\,d\theta = \ln|\tan\theta + \sec\theta| + C$) (10%)

《試題完》

1. 求下列積分：

(a) $\int x^2 \ln(x-1)\,dx$

$u = \ln(x-1), dv = x^2 dx, du = \frac{1}{x-1}dx, v = \frac{x^3}{3}$

$\int x^2 \ln(x-1)\,dx$

$= \frac{x^3}{3}\ln(x-1) - \frac{1}{3}\int \frac{x^3}{x-1}dx = \frac{x^3}{3}\ln(x-1) - \frac{1}{3}\int \left(x^2 + x + 1 + \frac{1}{x-1}\right)dx$

$= \frac{x^3}{3}\ln(x-1) - \frac{1}{3}\left(\frac{x^3}{3} + \frac{x^2}{2} + x + \ln(x-1)\right) + C$

$= \frac{x^3}{3}\ln(x-1) - \frac{x^3}{9} - \frac{x^2}{6} - \frac{x}{3} - \frac{1}{3}\ln(x-1) + C$

$= \frac{x^3-1}{3}\ln(x-1) - \frac{x^3}{9} - \frac{x^2}{6} - \frac{x}{3} + C$

(b) $\int_2^4 3^x\,dx$

$\int_2^4 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_2^4 = \frac{3^4}{\ln 3} - \frac{3^2}{\ln 3} = \frac{72}{\ln 3}$

(c) $\int (e^x + x)^2\,dx$

$\int (e^x + x)^2\,dx$

$= \int (e^{2x} + 2xe^x + x^2)\,dx = \frac{e^{2x}}{2} + 2xe^x - 2\int e^x\,dx + \frac{x^3}{3}$

$= \frac{e^{2x}}{2} + 2xe^x - 2e^x + \frac{x^3}{3} + C$

(d) $\int \sin^7 x \cos^2 x\,dx$

$\int \sin^7 x \cos^2 x\,dx$

$= \int (1-\cos^2 x)^3 \sin x \cos^2 x\,dx = \int (1-u^2)^3 u^2(-du) = \int (1-u^2)^3 u^2(-du)$

$= -\int (1 - 3u^2 + 3u^4 - u^6)\,u^2\,du = -\int (u^2 - 3u^4 + 3u^6 - u^8)du$

$= -\left(\frac{u^3}{3} - \frac{3u^5}{5} + \frac{3u^7}{7} - \frac{u^9}{9}\right) + C = -\frac{\cos^3 x}{3} + \frac{3\cos^5 x}{5} - \frac{3\cos^7 x}{7} + \frac{\cos^9 x}{9} + C$

(e) $\int \frac{1}{x^2\sqrt{9-4x^2}}\,dx$

$$x = \frac{3}{2}\sin\theta\,, dx = \frac{3}{2}\cos\theta\,d\theta$$

$$\int \frac{1}{x^2\sqrt{9-4x^2}}\,dx = \int \frac{1}{\frac{9}{4}\sin^2\theta\,\sqrt{9-9\sin^2\theta}}\left(\frac{3}{2}\cos\theta\,d\theta\right) = \frac{2}{9}\int \frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{2}{9}\int \csc^2\theta\,d\theta = -\frac{2}{9}\cot\theta + C = -\frac{\sqrt{9-4x^2}}{9x} + C$$

(f) $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

$$x = \frac{3}{2}\sin\theta\,, dx = \frac{3}{2}\cos\theta\,d\theta$$

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \int \frac{1}{\sqrt{9-9\sin^2\theta}}\left(\frac{3}{2}\cos\theta\,d\theta\right) = \frac{1}{2}\int d\theta = \frac{\theta}{2} + C = \frac{1}{2}\sin^{-1}\frac{2}{3}x + C$$

(g) $\int \frac{x^3-2x-4}{x^4+x^2}\,dx$

$$\int \frac{x^3-2x-4}{x^4+x^2}\,dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}\right)dx = \int \left(\frac{-2}{x} + \frac{-4}{x^2} + \frac{3x+4}{x^2+1}\right)dx$$

$$= \int \left(\frac{-2}{x} + \frac{-4}{x^2} + \frac{3x}{x^2+1} + \frac{4}{x^2+1}\right)dx = -2\ln|x| + 4x^{-1} + \frac{3}{2}\ln(x^2+1) + 4\tan^{-1}x + C$$

(h) $\int_1^\infty \frac{x+1}{x^3}\,dx$

$$\int_1^\infty \frac{x+1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \left(\frac{1}{x^2} + \frac{1}{x^3}\right)dx = \lim_{t\to\infty}\left[-\frac{1}{x} - \frac{1}{2x^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\left(-\frac{1}{t} - \frac{1}{2t^2}\right) - \left(-1 - \frac{1}{2}\right)\right] = \frac{3}{2}$$

(i) $\int x\cos x\,dx$

$u = x, dv = \cos x\,dx, du = dx, v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x - (-\cos x) + C = x\sin x + \cos x + C$$

(j) $\int_1^{29} \frac{1}{\sqrt[3]{x-2}}\,dx$

$$\int_1^{29} \frac{1}{\sqrt[3]{x-2}}\,dx = \int_1^2 \frac{1}{\sqrt[3]{x-2}}\,dx + \int_2^{29} \frac{1}{\sqrt[3]{x-2}}\,dx$$

$$= \lim_{t\to 2^-}\int_1^t \frac{1}{\sqrt[3]{x-2}}dx + \lim_{t\to 2^+}\int_t^{29} \frac{1}{\sqrt[3]{x-2}}dx$$

$$= \lim_{t\to 2^-}\left[\frac{3}{2}(x-2)^{\frac{2}{3}}\right]_1^t + \lim_{t\to 2^+}\left[\frac{3}{2}(x-2)^{\frac{2}{3}}\right]_t^{29} = -\frac{3}{2}+\frac{27}{2} = 12$$

2.

求交點座標 $y=(x-1)^2=2x+6 \rightarrow x^2-4x-5=0 \rightarrow (x+1)(x-5)=0 \rightarrow x=-1, 5 \rightarrow (-1,-2), (5,4)$

[解法1] $x_L=\frac{1}{2}y^2-3$, $x_R=y+1$ （左右法）

$$A=\int_{-2}^{4}(x_R-x_L)dy$$
$$=\int_{-2}^{4}[(y+1)-(\frac{1}{2}y^2-3)]dy$$
$$=\int_{-2}^{4}(-\frac{1}{2}y^2+y+4)dy$$
$$=[(-\frac{1}{2}\cdot\frac{1}{3}y^3)+\frac{1}{2}y^2+4y]_{-2}^{4}$$
$$=[(-\frac{1}{2}\cdot\frac{1}{3}4^3)+\frac{1}{2}4^2+4\cdot4]-[(-\frac{1}{2}\cdot\frac{1}{3}(-2)^3)+\frac{1}{2}(-2)^2+4(-2)]=18$$

[解法2] $A=A_1+A_2$ （上下法）

$y^2=2x+6 \rightarrow y=\pm\sqrt{2x+6}$

$$A_1=2\int_{-3}^{-1}\sqrt{2x+6}\,dx$$

令u=2x+6, $du=2dx$, 0≦u≦4 代入

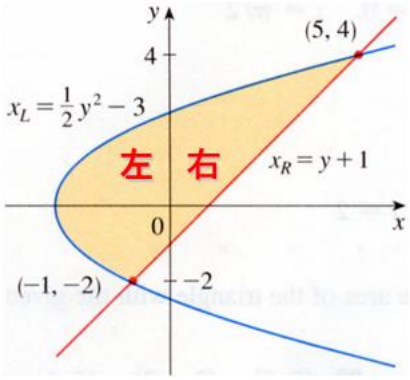
$$A_1=\int_0^4\sqrt{u}\,du$$
$$=[\frac{2}{3}u^{1.5}]_0^4$$
$$=\frac{16}{3}$$

$$A_2=\int_{-1}^{5}(\sqrt{2x+6}-x+1)dx$$

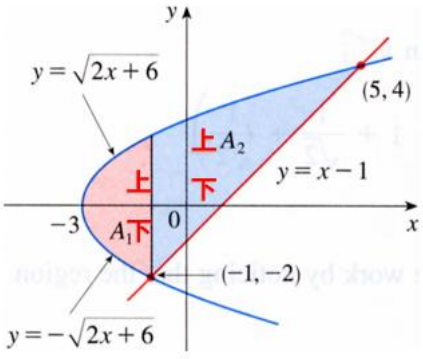
令u=2x+6, $du=2dx$, 4≦u≦16 代入

$$A_2=\frac{1}{2}[\int_4^{16}\sqrt{u}\,du+\frac{1}{2}\int_4^{16}(u-6)du]$$
$$=\frac{1}{2}[\frac{2}{3}u^{1.5}+\frac{1}{2}(\frac{1}{2}u^2-6u)]_4^{16}$$
$$=\frac{38}{3}$$

$$A=A_1+A_2=\frac{16}{3}+\frac{38}{3}=18$$

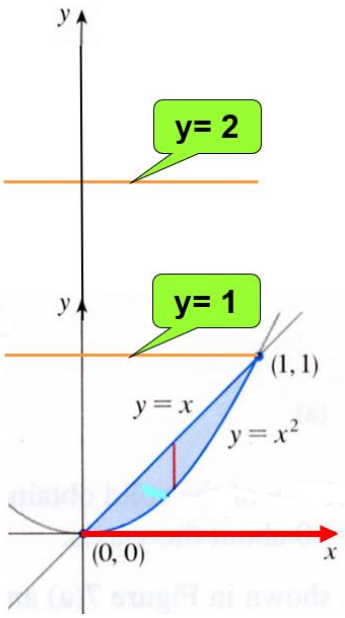


解法 1



解法 2

3.



求交點座標 $y=x=x^2 \rightarrow x(x-1)=0 \rightarrow x=0,1 \rightarrow (0,0),(1,1)$

判斷大小 $x=\frac{1}{2}$ 代入$y=x,\ y=x^2,\ \frac{1}{2} > (\frac{1}{2})^2$

使用Washer法 $A(x)=\pi(2-x^2)^2-\pi(2-x)^2=\pi(x^4-5x^2+4x)$

$$V=\int_0^1 A(x)dx$$
$$=\int_0^1 \pi(x^4-5x^2+4x)\,dx$$
$$=[\pi(\frac{1}{5}x^5-\frac{5}{3}x^3+2x^2)]_0^1$$
$$=\frac{8}{15}\pi$$

4.

弧長 $L=\int_0^1 \sqrt{1+(\frac{dx}{dy})^2}\,dy$ .....(1)

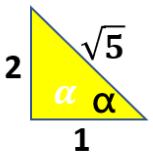
$x=y^2$ 兩邊微分 $dx=2y\,dy \rightarrow \frac{dx}{dy}=2y$ 代入(1)

$$L=\int_0^1 \sqrt{1+(\frac{dx}{dy})^2}\,dy=\int_0^1 \sqrt{1+4y^2}\,dy \quad .....(2)$$

令 $y=\frac{1}{2}\tan,\ dy=\frac{1}{2}sec^2\theta\,d\theta$

$y=0,\ \tan\theta=0,\ \theta=0$

$y=1,\ \tan\theta=2,\ \theta=\tan^{-1}2=\alpha$ 代入(2)



$$L=\int_0^\alpha \sqrt{1+\tan^2\theta}\cdot\frac{1}{2}sec^2\theta\,d\theta$$

$$=\frac{1}{2}\cdot\int_0^\alpha sec^3\theta\,d\theta \quad .....(3)$$

部分積分 $u=\sec\theta,\quad dv=\sec^2\theta d\theta$

$du=\sec\theta\tan\theta d\theta,\quad v=\tan\theta$ 代入(3)

$$L=\frac{1}{2}\cdot\int_0^\alpha sec^3\theta\,d\theta=\frac{1}{2}\{\sec\theta\tan\theta-\int_0^\alpha \sec\theta\tan^2\theta d\theta\}$$
$$L=\frac{1}{2}\cdot\int_0^\alpha sec^3\theta\,d\theta=\frac{1}{2}\{\sec\theta\tan\theta-\int_0^\alpha \sec\theta(\sec^2\theta-1)d\theta\}$$
$$=\frac{1}{2}\{\sec\theta\tan\theta-\int_0^\alpha \sec^3\theta d\theta+\int_0^\alpha \sec\theta d\theta\}$$

$$\int_0^\alpha sec^3\theta\,d\theta=\frac{1}{2}\{\sec\theta\tan\theta+\int_0^\alpha \sec\theta d\theta\} \quad .....(4)$$

已知 $\int\sec\theta d\theta=\ln|\tan\theta+\sec\theta|+C$ 代入(4)

$$\int_0^\alpha sec^3\theta\,d\theta=\frac{1}{2}\{\sec\theta\tan\theta+\ln|\tan\theta+\sec\theta|\}$$
$$=\frac{1}{2}\{\sqrt{5}\cdot 2+\ln|2+\sqrt{5}|\}$$

$$L=\frac{1}{2}\cdot\int_0^\alpha sec^3\theta\,d\theta=\frac{1}{4}\{2\sqrt{5}+\ln|2+\sqrt{5}|\}$$