國 立 雲 林 科 技 大 學

110 學年度第二學期工程學院會考試題

時間：10:10~12:00
科目：微積分
班級：

※本次測驗禁止使用電子計算機

1. 求下列積分：(6% × 10 = 60%)

(a) $\int_1^4 5^x\,dx$

(b) $\int \ln(x^{2x})dx$

(c) $\int \frac{dx}{\cot^4 x \cos^4 x}$

(d) $\int \frac{x^3+x}{x+1}dx$

(e) $\int \frac{1}{x^2\sqrt{64-x^2}}dx$

(f) $\int_2^\infty \frac{x^2+x+1}{x^4}dx$

(g) $\int \frac{dx}{\sqrt{x^2-6x+13}}$

(h) $\int \frac{\sin x}{\sqrt{4-\cos^2 x}}dx$

(i) $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}dx$

(j) $\int \frac{3x^3-2x-2}{x^2(x^2+1)}dx$

2. 求 $\lim_{x\to 0}\frac{\int_0^{x^2}(e^{t^2}+4)dt}{(\sin x)^2}$。 (10%)

3. 求 $y=x^2+2$ 與 $y=4-x$ 所圍區域繞x軸旋轉所得旋轉體的體積。 (10%)

4. 求 $y=\frac{1}{4}x^2-\frac{1}{2}\ln x$ 的圖形在 $1\le x\le 2$ 的弧長。 (10%)

5. 求拋物線 $y^2=x$ 和線性 $y+x=2$ 所包圍的面積。 (10%)

《試題完》

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1. Find the integral：(6% × 10 = 60%)

(a) $\int_1^4 5^x\,dx$

(b) $\int \ln(x^{2x})dx$

(c) $\int \frac{dx}{\cot^4 x \cos^4 x}$

(d) $\int \frac{x^3+x}{x+1}dx$

(e) $\int \frac{1}{x^2\sqrt{64-x^2}}dx$

(f) $\int_2^\infty \frac{x^2+x+1}{x^4}dx$

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(i) $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}dx$

(j) $\int \frac{3x^3-2x-2}{x^2(x^2+1)}dx$

2. Find $\lim_{x\to 0} \frac{\int_0^{x^2}(e^{t^2}+4)dt}{(\sin x)^2}$. (10%)

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 + 2$ and $y = 4 - x$ about the *x*-axis. (10%)

4. Find the arc length of $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ for $1 \le x \le 2$. (10%)

5. Find the area of the region bounded by the curves: $y^2 = x$ and $y + x = 2$. (10%)

《試題完》

1.

$(a)\ \int_1^4 5^x\,dx = \left[\frac{5^x}{ln\,5}\right]_1^4 = \frac{5^4}{ln\,5} - \frac{5^1}{ln5} = \frac{620}{ln5}$

$(b)\ \int ln(x^{2x})\,dx = \int x\,ln(x^2)\,dx$

$Let\ \ u = x^2\ \ ; du = 2xdx$

$= \frac{1}{2}\int ln\,u\,du = \frac{1}{2}ulnu - \frac{1}{2}u + C = \frac{1}{2}x^2 lnx^2 - \frac{1}{2}x^2 + C$

$(c)\ \int \frac{dx}{\cot^4 x \cos^4 x} = \int \tan^4 x \sec^4 x\,dx = \int \tan^4 x(1+\tan^2 x)\sec^2 x\,dx$

$Let\ \ u = \tan x\ \ ; du = \sec^2 x\,dx$

$= \int u^4(1+u^2)du = \frac{u^5}{5} + \frac{u^7}{7} + C = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

$(d)\ \int \frac{x^3+x}{x+1}dx = \int \left(x^2 - x + 2 - \frac{2}{x+1}\right)dx = \frac{x^3}{3} - \frac{x^2}{2} + 2x - 2ln|x+1| + C$

(e) $\int \frac{1}{x^2\sqrt{64-x^2}}dx$

$Let\ \ x = 8\sin\theta\ \ ; dx = 8\cos\theta\,d\theta$

$= \int \frac{8\cos\theta\,d\theta}{64\sin^2\theta\sqrt{64-64\sin^2\theta}} = \int \frac{8\cos\theta\,d\theta}{64\sin^2\theta\cdot 8\sqrt{1-\sin^2\theta}}$

$= \int \frac{8\cos\theta\,d\theta}{64\sin^2\theta\cdot 8\sqrt{\cos^2\theta}} = \frac{1}{64}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{64}\int \csc^2\theta\,d\theta$

$= -\frac{1}{64}\cot\theta + C = -\frac{\sqrt{64-x^2}}{64x} + C$

(f) $\int_2^\infty \frac{x^2+x+1}{x^4}dx = \lim_{t\to\infty}\int_2^t (x^{-2}+x^{-3}+x^{-4})dx$

$= \lim_{t\to\infty}\left[-\frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3}\right]_2^t = \lim_{t\to\infty}\left[\left(-\frac{1}{t} - \frac{1}{2t^2} - \frac{1}{3t^3}\right) - \left(-\frac{1}{2} - \frac{1}{8} - \frac{1}{24}\right)\right]$

$= 0 + \frac{2}{3} = \frac{2}{3}$

(g) $\int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dx}{\sqrt{(x-3)^2+4}} \xrightarrow[dx=2\sec^2\theta d\theta]{x-3=2\tan\theta} \int \frac{2\sec^2\theta}{2\sec\theta} d\theta$

$= \int \sec\theta \, d\theta$

$\Rightarrow \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{x^2-6x+13}}{2} + \frac{x-3}{2}\right| + C'$

$= \ln\left|\sqrt{x^2-6x+13} + x - 3\right| + C$

(h) $\int \frac{\sin x}{\sqrt{4-\cos^2 x}} dx$ =?

令 $\cos x = u \Rightarrow -\sin x \, dx = du$

原式 $= \int \frac{-1}{\sqrt{2^2-u^2}} du = -\sin^{-1}\left(\frac{u}{2}\right) + C = -\sin^{-1}\left(\frac{\cos x}{2}\right) + C$

(i) $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx = \int_0^{1^-} \frac{1}{(x-1)^{\frac{2}{3}}} dx + \int_{1^+}^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx$

$= \lim_{c\to 1^-} \int_0^c \frac{1}{(x-1)^{\frac{2}{3}}} dx + \lim_{c\to 1^+} \int_c^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx$

$= \lim_{c\to 1^-} 3(x-1)^{\frac{1}{3}}|_{x=0}^{x=c} + \lim_{c\to 1^+} 3(x-1)^{\frac{1}{3}}|_{x=c}^{x=3}$

$= \lim_{c\to 1^-} [3(c-1)^{\frac{1}{3}} + 3] + \lim_{c\to 1^+} [3\cdot 2^{\frac{1}{3}} - 3(c-1)^{\frac{1}{3}}]$

$= 3 + 3\sqrt[3]{2}$

(j) $\frac{3x^3-2x-2}{x^2(x^2+1)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{5x+2}{x^2+1}$

$\int \frac{3x^3-2x-2}{x^2(x^2+1)} dx = \int \frac{-2}{x} + \frac{-2}{x^2} + \frac{5x+2}{x^2+1} dx$

$= -2\,ln|x| + 2x^{-1} + \frac{5}{2} ln(x^2+1) + 2\,tan^{-1} x + c$

2. $\lim_{x\to 0}\frac{\int_0^{x^2}(e^{t^2}+4)dt}{(\sin x)^2}=?$

Sol: $\because \frac{0}{0}$ 型，使用$L'Hospital\ rule$及萊布尼茲法則並用

$$\lim_{x\to 0}\frac{\int_0^{x^2}(e^{t^2}+4)dt}{(\sin x)^2}=\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^{x^2}(e^{t^2}+4)\,dt}{\frac{d}{dx}(\sin x)^2}=\lim_{x\to 0}\frac{(e^{x^4}+4)2x}{2\sin x\cos x}$$

$$=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\left(\frac{e^{x^4}+4}{\cos x}\right)=5$$

3. 兩曲線交點為(1,3) (-2,6)

$$A(x)=\pi(4-x)^2-\pi\left(x^2+2\right)^2=\pi(12-8x-3x^2-x^4)$$

$$\mathrm{V}=\int_{-2}^{1}A(x)\,dx=\pi\int_{-2}^{1}(12-8x-3x^2-x^4)\,dx=\pi\left[12x-4x^2-x^3-\frac{x^5}{5}\right]_{-2}^{1}$$

$$=\pi\left(\frac{34}{5}-\frac{-128}{5}\right)=\frac{162}{5}\pi$$

4. $y=\frac{1}{4}x^2-\frac{1}{2}\ln x,\ \frac{dy}{dx}=\frac{1}{2}x-\frac{1}{2x}$

$$\int_1^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_1^2\sqrt{1+\left(\frac{1}{2}x-\frac{1}{2x}\right)^2}\,dx=\int_1^2\sqrt{1+\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}}\,dx=\int_1^2\sqrt{\frac{1}{2}+\frac{x^2}{4}+\frac{1}{4x^2}}\,dx=$$

$$\int_1^2\sqrt{\left(\frac{1}{2}x+\frac{1}{2x}\right)^2}\,dx=\frac{1}{4}x^2+\frac{1}{2}\ln|x|\Big|_1^2=\frac{3}{4}+\frac{1}{2}\ln 2$$

5. $y^2=x$ 帶入 $y+x=2$ 解聯立得交點 (1,1)和(4,-2)

面積 $=\int_{-2}^{1}|x_2-x_1|\,dy$ …… $\{\because |x_2-x_1|=x_2-x_1, x_2>x_1\}$

$$=\int_{-2}^{1}\{(2-y)-y^2\,dy\}=\left[2y-\frac{y^2}{2}-\frac{y^3}{3}\right]_{-2}^{1}=\left[2-\frac{1}{2}-\frac{1}{3}\right]-\left[-4-2+\frac{8}{3}\right]=\frac{9}{2}$$

