國 立 雲 林 科 技 大 學

108 學年度第二學期工程學院會考試題

時間：10:10~12:00
科目：微積分 (Calculus)
班級：

※本次測驗禁止使用電子計算機.

1. 求下列不定積分： $(7\% \times 6 = 42\%)$

(a) $\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8} d\theta$

(b) $\int \sin^3 x \cos^2 x \, dx$

(c) $\int (\ln x)^2 dx$

(d) $\int \frac{1}{x} \arcsin(\ln x) dx$

(e) $\int x \ln(x+1) dx$

(f) $\int \frac{x}{1+x^4} dx$

2. 求 $\int_0^3 \frac{x^3}{\sqrt{9+x^2}} dx$ 。 (8%)

3. 分別計算 $y = x^3, y = 0, x = 2$ 所圍區域分別繞 (a) x 軸 與 (b) 直線 $x = 4$ 旋轉所得旋轉體的體積。 (10%)

4. 畫出 $f(x) = 2\sin x$, $g(x) = \tan x$ 在 $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$ 範圍內所圍區域的圖形，並計算此區域的面積。 (10%)

5. 計算曲線 $y = \frac{x^3}{6} + \frac{1}{2x}$ 在 $[1, 2]$ 區間之弧長。 (10%)

6. 計算 $y = \sqrt{x}, y = -\frac{1}{2}x + 4, x = 0$ 與 $x = 8$ 所圍區域繞 x 軸旋轉所得旋轉體的體積。 (10%)

7. 求 $\lim_{x \to 0^+} (1+x)^{1/x}$ 。 (10%)

《試題完》

1. (a)

$$\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\,d\theta$$

Let $u = \sin\theta$,

then $du = \cos\theta\, d\theta$

$$\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\,d\theta$$

$$= \int \frac{6}{u^2 - 2u - 8}\,du$$

$$= \int \left(\frac{a}{u-4} + \frac{b}{u+2} \right) du$$

$$\begin{cases} a + b = 0 & (1) \\ 2a - 4b = 6 & (2) \end{cases}$$

$$\Rightarrow (1) \times 4 + (2)$$

$$\begin{cases} a = 1 \\ b = -1 \end{cases}$$

$$\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\,d\theta$$

$$= \int \left(\frac{1}{u-4} + \frac{-1}{u+2} \right) du$$

$$= \ln|u - 4| - \ln|u + 2| + c$$

$$= \ln\left|\frac{u-4}{u+2}\right| + c$$

$$= \ln\left|\frac{\sin\theta - 4}{\sin\theta + 2}\right| + c$$

1. (b)

$$\int \sin^3 x \cos^2 x\, dx$$

Let $u = \cos x$,

then $du = -\sin x\, dx$

$$\int \sin^3 x \cos^2 x\, dx$$

$$= \int \left[-(1 - u^2)u^2 \right] du$$

$$= \int \left(u^4 - u^2 \right) du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$$

1. (c)

$$\int (\ln x)^2 dx$$

Let $u = (\ln x)^2, dv = dx,$

then $du = 2\ln x \cdot \frac{1}{x} dx, v = x$

$$\int (\ln x)^2 dx$$

$$= x(\ln x)^2 - \int x \cdot 2\ln x \cdot \frac{1}{x} dx$$

$$= x(\ln x)^2 - 2\int \ln x dx$$

Let $u = \ln x, dv = dx,$

then $du = \frac{1}{x} dx, v = x$

$$\int \ln x dx$$

$$= x\ln x - \int x \cdot \frac{1}{x} dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + c$$

Hence

$$\int (\ln x)^2 dx$$

$$= x(\ln x)^2 - 2(x\ln x - x + c)$$

$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

1. (d)

$$\int \frac{1}{x} \arcsin(\ln x) dx$$

Let $w = \ln x,$

then $dw = \frac{1}{x} dx$

$$\int \frac{1}{x} \arcsin(\ln x) dx$$

$$= \int \arcsin w dw$$

Let $u = \arcsin w, \quad dv = dw,$

then $du = \frac{dw}{\sqrt{1-w^2}}, \quad v = w$

$$\int \frac{1}{x} \arcsin(\ln x) dx$$

$$= w\arcsin w - \int \frac{w}{\sqrt{1-w^2}} dw$$

$$= w\arcsin w + \frac{1}{2}\int \frac{d(1-w^2)}{\sqrt{1-w^2}}$$

$$= w\arcsin w + \frac{1}{2 \times \frac{1}{2}} \sqrt{1-w^2} + c$$

$$= w\arcsin w + \sqrt{1-w^2} + c$$

$$= (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2} + c$$

1. (e)

$$\int x\ln(x+1)dx$$

Let $u=\ln(x+1),\quad dv=xdx$,

then $du=\dfrac{dx}{x+1},\quad v=\dfrac{1}{2}x^2$

$$\int x\ln(x+1)dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\int\frac{x^2}{x+1}dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\left(\frac{1}{2}x^2-x+\ln(x+1)\right)+c$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{4}x^2+\frac{1}{2}x-\frac{1}{2}\ln(x+1)+c$$

$$=-\frac{1}{4}x^2+\frac{1}{2}x+\frac{x^2-1}{2}\ln(x+1)+c$$

1. (f)

$$\int\frac{x}{1+x^4}dx$$

Let $u=x^2$, then $du=2x\,dx$

$$\int\frac{x}{1+x^4}dx$$

$$=\int\frac{\frac{1}{2}du}{1+u^2}$$

$$=\frac{1}{2}\arctan u+c$$

$$=\frac{1}{2}\arctan x^2+c$$

2.

$$\int_0^3 \frac{x^3}{\sqrt{9+x^2}}dx$$

Let $x = 3\tan u$,

then $dx = 3\sec^2 u\, du$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow \tan u = 1 \Rightarrow u = \frac{\pi}{4}$

In this region, tan$u$ and sec$u$ are positive.

$$\int_0^3 \frac{x^3}{\sqrt{9+x^2}}dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{27\tan^3 u \cdot 3\sec^2 u}{3\sec u}du$$

$$= 27\int_0^{\frac{\pi}{4}} \tan^3 u \cdot \sec u du$$

$$= 27\int_0^{\frac{\pi}{4}} \tan^2 u\, d(\sec u)$$

$$= 27\int_0^{\frac{\pi}{4}} (\sec^2 u - 1)\, d(\sec u)$$

$$= 27\left[\frac{1}{3}\sec^3 u - \sec u\right]_0^{\frac{\pi}{4}}$$

$$= 27\left[\frac{1}{3}(2\sqrt{2}-1) - (\sqrt{2}-1)\right]$$

$$= 27 \times \frac{-\sqrt{2}+2}{3}$$

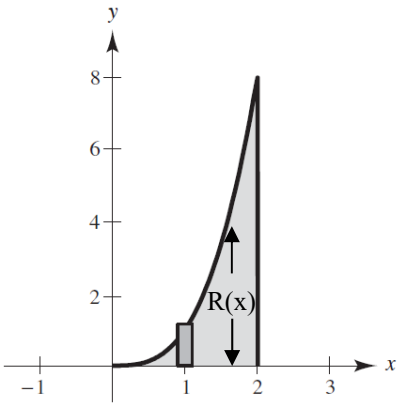
$$= 18 - 9\sqrt{2}$$

3. (a)

$R(x) = x^3, r(x) = 0$

$V = \pi\int_0^2 [R^2(x) - r^2(x)]dx = \pi\int_0^2 x^6 dx$

$= \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$

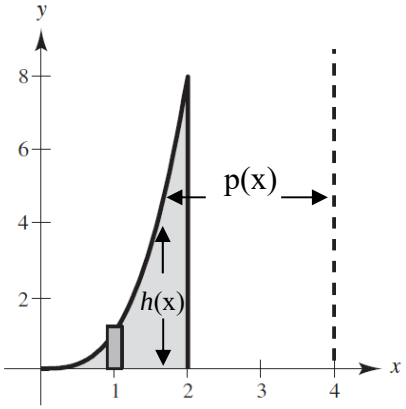


3.(b)

$p(x) = 4 - x, h(x) = x^3$

$V = 2\pi\int_0^2 p(x)h(x)dx = 2\pi\int_0^2 (4-x)x^3 dx$

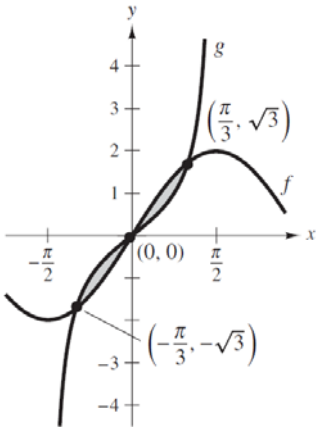
$= 2\pi\int_0^2 (4x^3 - x^4)dx = 2\pi\left[x^4 - \frac{x^5}{5}\right]_0^2 = \frac{96\pi}{5}$



4.

$f(x) = 2\sin x, \quad g(x) = \tan x$

(0.1)



$A = 2\int_0^{\pi/3} [f(x) - g(x)]dx$

$= 2\int_0^{\pi/3} [2\sin x - \tan x]dx$

$= 2[-2\cos x + \ln|\cos x|]_0^{\pi/3}$

$= 2(1 - \ln 2) \approx 0.614$

5.

$y = \frac{x^3}{6} + \frac{1}{2x}, \quad y' = \frac{x^2}{2} - \frac{1}{2x^2}, 1 \le x \le 2$

$1 + (y')^2 = 1 + \frac{1}{4}(x^4 - 2 + \frac{1}{x^4}) = \frac{1}{4}(x^4 + 2 + \frac{1}{x^4})^2$

$= \left[\frac{1}{2}(x^2 + \frac{1}{x^2})\right]^2$

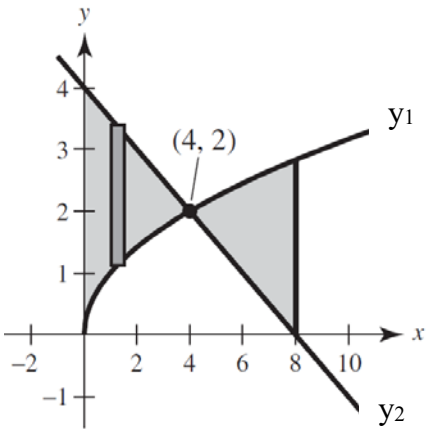
$S = \int_1^2 \sqrt{1 + (y')^2}\,dx$

$= \int_1^2 \frac{1}{2}(x^2 + \frac{1}{x^2})dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_1^2$

$= \frac{1}{2}\left[(\frac{8}{3} - \frac{1}{2}) - (\frac{1}{3} - 1)\right] = \frac{17}{12}$

6.

$y_1 = \sqrt{x},\ y_2 = -\frac{1}{2}x + 4$



$\sqrt{x} = -\frac{1}{2}x + 4$

$x = -\frac{1}{4}x^2 - 4x + 16$

$0 = x^2 - 20x + 16$

$0 = (x-4)(x-16)$

The curves intersect at (4,2) (x=16 is an extraneous root)

$$V = \pi\int_0^4 \left[(4-\frac{1}{2}x)^2 - (\sqrt{x})^2 \right] dx + \pi\int_4^8 \left[(\sqrt{x})^2 - (4-\frac{1}{2}x)^2 \right] dx$$

$$= \pi\int_0^4 (\frac{x^2}{4} - 5x + 16)dx + \pi\int_4^8 (-\frac{x^2}{4} + 5x - 16)dx$$

$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x \right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x \right]_4^8$$

$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$

7.

indeterminate form: $\lim_{x\to 0^+}(1+x)^{1/x} = 1^\infty$

Let $y = \lim_{x\to 0^+}(1+x)^{1/x}$

$$\ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = e$

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108 學年度第二學期工程學院會考試題

時間：10:10~12:00

科目：微積分 (Calculus)

班級：

※Calculator is not allowed for this test.

1. Calculate $(7\% \times 6 = 42\%)$

(a) $\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8} d\theta$

(b) $\int \sin^3 x \cos^2 x \, dx$

(c) $\int (\ln x)^2 dx$

(d) $\int \frac{1}{x} \arcsin(\ln x) dx$

(e) $\int x \ln(x+1) dx$

(f) $\int \frac{x}{1+x^4} dx$

2. Evaluate $\int_0^3 \frac{x^3}{\sqrt{9+x^2}} dx$ (8%)

3. Find the volumes of the solids generated by revolving the region bounded by the graphs of $y = x^3, y = 0, x = 2$ about the (a) x-axis, (b) line $x = 4$. (10%)

4. Sketch the region bounded by the graphs of $f(x) = 2\sin x$, $g(x) = \tan x$ and $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$, and find the area of the region. (10%)

5. Evaluate the definite integral for the arc length of the curve of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval [1, 2]. (10%)

6. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x}, y = -\frac{1}{2}x + 4, x = 0$, and $x = 8$ about the x-axis. (10%)

7. Evaluate $\lim_{x \to 0^+} (1+x)^{1/x}$. (10%)

《END of the SHEET》

1. (a)

$$\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\,d\theta$$

Let $u = \sin\theta$,

then $du = \cos\theta\, d\theta$

$$\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\,d\theta$$

$$= \int \frac{6}{u^2 - 2u - 8}\,du$$

$$= \int \left(\frac{a}{u-4} + \frac{b}{u+2} \right) du$$

$$\begin{cases} a + b = 0 & (1) \\ 2a - 4b = 6 & (2) \end{cases}$$

$$\Rightarrow (1) \times 4 + (2)$$

$$\begin{cases} a = 1 \\ b = -1 \end{cases}$$

$$\int \frac{6\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\,d\theta$$

$$= \int \left(\frac{1}{u-4} + \frac{-1}{u+2} \right) du$$

$$= \ln|u - 4| - \ln|u + 2| + c$$

$$= \ln\left|\frac{u-4}{u+2}\right| + c$$

$$= \ln\left|\frac{\sin\theta - 4}{\sin\theta + 2}\right| + c$$

1. (b)

$$\int \sin^3 x \cos^2 x\, dx$$

Let $u = \cos x$,

then $du = -\sin x\, dx$

$$\int \sin^3 x \cos^2 x\, dx$$

$$= \int \left[-(1 - u^2)u^2 \right] du$$

$$= \int \left(u^4 - u^2 \right) du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$$

1. (c)

$$\int (\ln x)^2 dx$$

Let $u = (\ln x)^2, dv = dx,$

then $du = 2\ln x \cdot \frac{1}{x} dx, v = x$

$$\int (\ln x)^2 dx$$

$$= x(\ln x)^2 - \int x \cdot 2\ln x \cdot \frac{1}{x} dx$$

$$= x(\ln x)^2 - 2\int \ln x dx$$

Let $u = \ln x, dv = dx,$

then $du = \frac{1}{x} dx, v = x$

$$\int \ln x dx$$

$$= x\ln x - \int x \cdot \frac{1}{x} dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + c$$

Hence

$$\int (\ln x)^2 dx$$

$$= x(\ln x)^2 - 2(x\ln x - x + c)$$

$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

1. (d)

$$\int \frac{1}{x} \arcsin(\ln x) dx$$

Let $w = \ln x,$

then $dw = \frac{1}{x} dx$

$$\int \frac{1}{x} \arcsin(\ln x) dx$$

$$= \int \arcsin w dw$$

Let $u = \arcsin w, \quad dv = dw,$

then $du = \frac{dw}{\sqrt{1-w^2}}, \quad v = w$

$$\int \frac{1}{x} \arcsin(\ln x) dx$$

$$= w\arcsin w - \int \frac{w}{\sqrt{1-w^2}} dw$$

$$= w\arcsin w + \frac{1}{2}\int \frac{d(1-w^2)}{\sqrt{1-w^2}}$$

$$= w\arcsin w + \frac{1}{2 \times \frac{1}{2}} \sqrt{1-w^2} + c$$

$$= w\arcsin w + \sqrt{1-w^2} + c$$

$$= (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2} + c$$

1. (e)

$$\int x\ln(x+1)dx$$

Let $u=\ln(x+1),\quad dv=xdx$,

then $du=\dfrac{dx}{x+1},\quad v=\dfrac{1}{2}x^2$

$$\int x\ln(x+1)dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\int\frac{x^2}{x+1}dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\left(\frac{1}{2}x^2-x+\ln(x+1)\right)+c$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{4}x^2+\frac{1}{2}x-\frac{1}{2}\ln(x+1)+c$$

$$=-\frac{1}{4}x^2+\frac{1}{2}x+\frac{x^2-1}{2}\ln(x+1)+c$$

1. (f)

$$\int\frac{x}{1+x^4}dx$$

Let $u=x^2$, then $du=2x\,dx$

$$\int\frac{x}{1+x^4}dx$$

$$=\int\frac{\frac{1}{2}du}{1+u^2}$$

$$=\frac{1}{2}\arctan u+c$$

$$=\frac{1}{2}\arctan x^2+c$$

2.

$$\int_0^3 \frac{x^3}{\sqrt{9+x^2}}dx$$

Let $x = 3\tan u$,

then $dx = 3\sec^2 u\, du$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow \tan u = 1 \Rightarrow u = \frac{\pi}{4}$

In this region, tan$u$ and sec$u$ are positive.

$$\int_0^3 \frac{x^3}{\sqrt{9+x^2}}dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{27\tan^3 u \cdot 3\sec^2 u}{3\sec u}du$$

$$= 27\int_0^{\frac{\pi}{4}} \tan^3 u \cdot \sec u du$$

$$= 27\int_0^{\frac{\pi}{4}} \tan^2 u\, d(\sec u)$$

$$= 27\int_0^{\frac{\pi}{4}} (\sec^2 u - 1)\, d(\sec u)$$

$$= 27\left[\frac{1}{3}\sec^3 u - \sec u\right]_0^{\frac{\pi}{4}}$$

$$= 27\left[\frac{1}{3}(2\sqrt{2}-1) - (\sqrt{2}-1)\right]$$

$$= 27 \times \frac{-\sqrt{2}+2}{3}$$

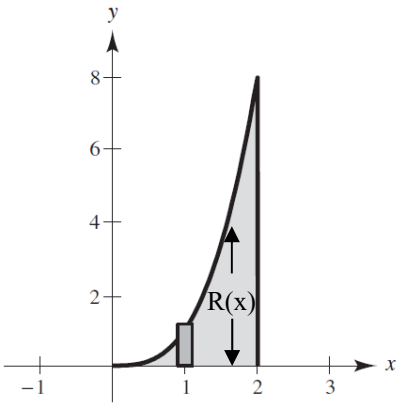
$$= 18 - 9\sqrt{2}$$

3. (a)

$R(x) = x^3, r(x) = 0$

$V = \pi\int_0^2 [R^2(x) - r^2(x)]dx = \pi\int_0^2 x^6 dx$

$= \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$

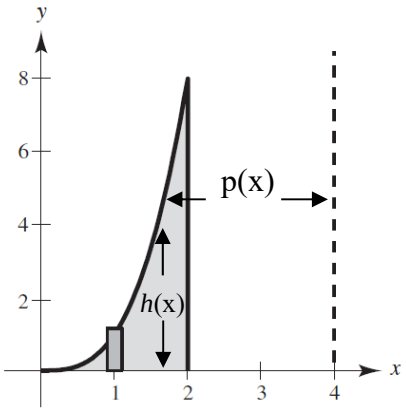


3.(b)

$p(x) = 4 - x, h(x) = x^3$

$V = 2\pi\int_0^2 p(x)h(x)dx = 2\pi\int_0^2 (4-x)x^3 dx$

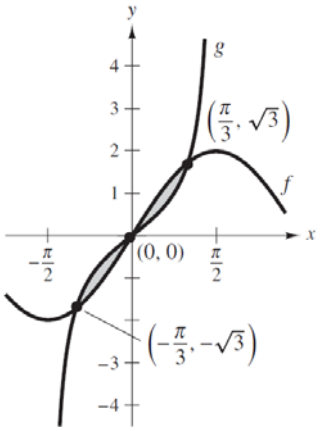
$= 2\pi\int_0^2 (4x^3 - x^4)dx = 2\pi\left[x^4 - \frac{x^5}{5}\right]_0^2 = \frac{96\pi}{5}$



4.

$f(x) = 2\sin x, \quad g(x) = \tan x$

(1.1)



$A = 2\int_0^{\pi/3} [f(x) - g(x)]dx$

$= 2\int_0^{\pi/3} [2\sin x - \tan x]dx$

$= 2[-2\cos x + \ln|\cos x|]_0^{\pi/3}$

$= 2(1 - \ln 2) \approx 0.614$

5.

$y = \frac{x^3}{6} + \frac{1}{2x}, \quad y' = \frac{x^2}{2} - \frac{1}{2x^2}, 1 \le x \le 2$

$1 + (y')^2 = 1 + \frac{1}{4}(x^4 - 2 + \frac{1}{x^4}) = \frac{1}{4}(x^4 + 2 + \frac{1}{x^4})^2$

$= \left[\frac{1}{2}(x^2 + \frac{1}{x^2})\right]^2$

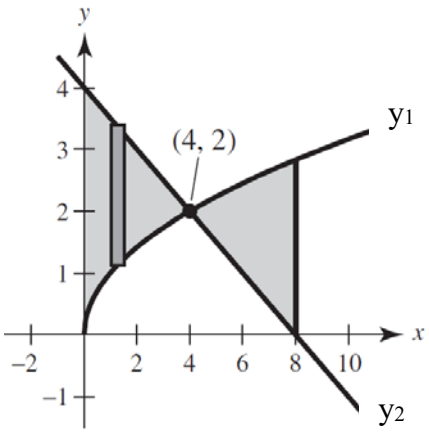
$S = \int_1^2 \sqrt{1 + (y')^2}\,dx$

$= \int_1^2 \frac{1}{2}(x^2 + \frac{1}{x^2})dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_1^2$

$= \frac{1}{2}\left[(\frac{8}{3} - \frac{1}{2}) - (\frac{1}{3} - 1)\right] = \frac{17}{12}$

6.

$y_1 = \sqrt{x},\ y_2 = -\frac{1}{2}x + 4$



$\sqrt{x} = -\frac{1}{2}x + 4$

$x = -\frac{1}{4}x^2 - 4x + 16$

$0 = x^2 - 20x + 16$

$0 = (x-4)(x-16)$

The curves intersect at (4,2) (x=16 is an extraneous root)

$$V = \pi\int_0^4 \left[(4-\frac{1}{2}x)^2 - (\sqrt{x})^2 \right] dx + \pi\int_4^8 \left[(\sqrt{x})^2 - (4-\frac{1}{2}x)^2 \right] dx$$

$$= \pi\int_0^4 (\frac{x^2}{4} - 5x + 16)dx + \pi\int_4^8 (-\frac{x^2}{4} + 5x - 16)dx$$

$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x \right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x \right]_4^8$$

$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$

7.

Indeterminate form: $\lim_{x\to 0^+}(1+x)^{1/x} = 1^\infty$

Let $y = \lim_{x\to 0^+}(1+x)^{1/x}$

$$\ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = e$