國 立 雲 林 科 技 大 學

106 學年度第二學期工學院會考試題

時間：10:10~12:00
科目：微積分
班級：

1. 請計算 $\int_0^1 x^2 10^{(x^3+1)}\,dx$ (7%)

2. 請計算 $\int x^2 \ln(x+1)\,dx$ (提示：分部積分) (7%)

3. 請計算 $\int_0^1 x^2 \arctan x^3\,dx$ (提示：分部積分) (7%)

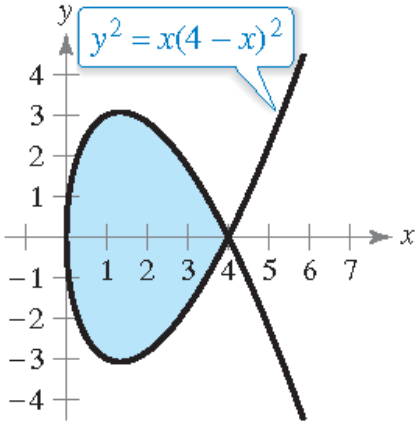
4. 請計算 $\int \tan^5 x \sec^3 x\,dx$ (7%)

5. 請計算 $\int \frac{dx}{\sqrt{x^2+4x+13}}$ (提示：先配方，再變數變換為三角函數) (7%)

6. 請計算 $\int \frac{x^2}{\sqrt{9-x^2}}dx$ (提示：變數變換為三角函數) (7%)

7. 請計算 $\int \frac{2x^2-x+3}{x(x-1)^2}dx$ (提示：化為部分分式) (8%)

8. 考慮 $y^2=x(4-x)^2$ 的圖形所圍的區域，如右圖。



(a) 求此區域面積。(5%)

(b) 求此區域對 $x=4$ 旋轉所得旋轉體的體積。(10%)

(c) 求此區域對 $y=-4$ 旋轉所得旋轉體的體積。(10%)

9. 求 $y=\frac{1}{2}(e^x+e^{-x})$ 的圖形在 $0\le x\le 2$ 的弧長。(7%)

10. 求 $\lim_{x\to 0^+}(\sin x)^x$。 (8%)

11. 求瑕積分 $\int_0^\infty \frac{2}{\sqrt{x}\,(x+4)}dx$。 (10%)

106-2 微積分會考解答

1.

$$I_1=\int_0^1 x^2\,10^{x^3+1}dx=\frac{1}{3}\int_{x=0}^{x=1}\exp[\ln(10^{x^3+1})]d(x^3+1)=\frac{1}{3}\int_{x=0}^{x=1}\exp[(x^3+1)\ln(10)]d(x^3+1)$$

$$=\frac{1}{3\ln 10}\left[10^{x^3+1}\right]_{x=0}^{x=1}=\frac{30}{\ln 10}.\quad \#\text{QED}$$

2. $u=\ln(x+1),\ dv=x^2dx,\ du=\frac{1}{x+1}dx,\ v=\frac{x^3}{3}$

$$I_2=\int x^2\ln(x+1)dx=\frac{x^3}{3}\ln(x+1)-\frac{1}{3}\int\frac{x^3}{x+1}dx=\frac{x^3}{3}\ln(x+1)-\frac{1}{3}\int\left(x^2-x+1-\frac{1}{x+1}\right)dx$$

$$=\frac{x^3}{3}\ln(x+1)+\frac{1}{3}\ln(x+1)-\frac{1}{9}x^3+\frac{1}{6}x^2-\frac{1}{3}x+C$$

$$=\frac{1}{3}(x^3+1)\ln(x+1)-\frac{1}{9}x^3+\frac{1}{6}x^2-\frac{1}{3}x+C.\quad \#\text{QED}$$

(Check: $\frac{dI_2}{dx}=\frac{1}{3}\frac{x^3+1}{x+1}+x^2\ln(x+1)-\frac{1}{3}(x^2-x+1)=x^2\ln(x+1)$)

3. $u=\tan^{-1}x^3,\ dv=x^2dx,\ du=\frac{3x^2}{x^6+1}dx,\ v=\frac{x^3}{3}$

$$I_3=\int_0^1 x^2\arctan x^3dx=\left[\frac{1}{3}x^3\tan^{-1}x^3\right]_0^1-\int_0^1\frac{x^5}{x^6+1}dx=\frac{\pi}{12}-\left[\frac{1}{6}\ln(x^6+1)\right]_0^1$$

$$=\frac{\pi}{12}-\frac{\ln 2}{6}.\quad \#\text{QED}$$

4.

$$I_4=\int\sin x\ \sin(5x)\cos x\cos(5x)dx=\frac{1}{4}\int\sin(2x)\sin(10x)dx$$

$$=\frac{1}{8}\int[\cos(2x-10x)-\cos(2x+10x)]dx=\frac{1}{64}\sin(8x)-\frac{1}{96}\sin(12x)+C.\quad \#\text{QED}$$

Check:

$$\frac{dI_4}{dx}=\frac{1}{8}[\cos(8x)-\cos(12x)]$$

$$=\frac{1}{8}[\cos(2x)\cos(10x)+\sin(2x)\sin(10x)-\cos(2x)\cos(10x)+\sin(2x)\sin(10x)]$$

$$=\frac{1}{4}\sin(2x)\sin(10x)=\sin x\cos x\sin(5x)\cos(5x)$$

5.

$$I_5=\int\tan^5 x\sec^3 xdx=\int\tan^4 x\sec^2 x\,d(\sec x)=\int(\sec^2 x-1)^2\sec^2 x\,d(\sec x)$$

$$=\int(\sec^6 x-2\sec^4 x+\sec^2 x)\,d(\sec x)=\frac{\sec^7 x}{7}-\frac{2\sec^5 x}{5}+\frac{\sec^3 x}{3}+C.\quad \#\text{QED}$$

6.

$$I_6 = \int \frac{dx}{\sqrt{x^2+4x+13}} = \int \frac{d(x+2)}{\sqrt{(x+2)^2+9}} \underset{\substack{x+2=3\tan\theta \\ d(x+2)=3\sec^2\theta d\theta}}{=} \int \frac{3\sec^2\theta}{3\sec\theta} d\theta$$

$$= \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + K = \ln\left|\frac{\sqrt{x^2+4x+13}}{3} + \frac{x+2}{3}\right| + K$$

$$= \ln\left|\sqrt{x^2+4x+13} + x + 2\right| + C. \quad \#\text{QED}$$

7.

$$I_7 = \int \frac{x^2}{\sqrt{9-x^2}} dx \underset{\substack{x=3\sin\theta \\ dx=3\cos\theta d\theta}}{=} \int \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta d\theta = \frac{9}{2}\int [1-\cos(2\theta)] d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C = \frac{9}{2}[\theta - \sin\theta\cos\theta] + C$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right] + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C. \quad \#\text{QED}$$

Check:

$$\frac{dI_7}{dx} = \frac{9}{2}\frac{1/3}{\sqrt{1-(x/3)^2}} - \frac{1}{2}\sqrt{9-x^2} + \frac{1}{2}x\frac{x}{\sqrt{9-x^2}} = \frac{9}{2}\frac{1}{\sqrt{9-x^2}} - \frac{1}{2}\sqrt{9-x^2} + \frac{1}{2}\frac{x^2}{\sqrt{9-x^2}}$$

$$= \frac{1}{2}\frac{9-(9-x^2)+x^2}{\sqrt{9-x^2}} = \frac{x^2}{\sqrt{9-x^2}}$$

8.

$$I_8 = \int \frac{2x^2-x+3}{x^3-2x^2+x} dx = \int \left[\frac{3}{x} - \frac{1}{x-1} + \frac{4}{(x-1)^2}\right] dx$$

$$= 3\ln|x| - \ln|x-1| - \frac{4}{x-1} + C = \ln\left|\frac{x^3}{x-1}\right| - \frac{4}{x-1} + C. \quad \#\text{QED}$$

9.

$$y^2 = x(4-x)^2 \;\rightarrow\; y = \pm\sqrt{x}\,(4-x)$$

(a)

$$A = \int_0^4 2\sqrt{x}(4-x)dx = \int_0^4 (8x^{1/2} - 2x^{3/2})dx = \left[\frac{8x^{3/2}}{3/2} - \frac{2x^{5/2}}{5/2}\right]_0^4 = \frac{16}{3}(4)^{3/2} - \frac{4}{5}(4)^{5/2}$$

$$= \frac{256}{15} \approx 17.067$$

(b)

$$V = \int_0^4 2\pi(4-x)\,2\sqrt{x}(4-x)\,dx = 4\pi\int_0^4 \sqrt{x}(x^2-8x+16)\,dx = 4\pi\int_0^4 (x^{5/2}-8x^{3/2}+16x^{1/2})\,dx$$

$$= 4\pi\left[\frac{x^{7/2}}{7/2}-\frac{8x^{5/2}}{5/2}+\frac{16x^{3/2}}{3/2}\right]_0^4 = 4\pi\left[\frac{2}{7}(4)^{7/2}-\frac{16}{5}(4)^{5/2}+\frac{32}{3}(4)^{3/2}\right] = \frac{8192}{105}\pi \approx 245.104$$

(c)

$$V = \int_0^4 \pi\left\{\left[\sqrt{x}(4-x)+4\right]^2-\left[-\sqrt{x}(4-x)+4\right]^2\right\}dx = \pi\int_0^4 16\sqrt{x}(4-x)\,dx$$

$$= 16\pi\int_0^4 (4x^{1/2}-x^{3/2})\,dx = 16\pi\left[\frac{4x^{3/2}}{3/2}-\frac{x^{5/2}}{5/2}\right]_0^4 = 16\pi\left[\frac{8}{3}(4)^{3/2}-\frac{2}{5}(4)^{5/2}\right]$$

$$= \frac{2048}{15}\pi \approx 428.932$$

10.

$$y = \frac{e^x+e^{-x}}{2},\quad y' = \frac{e^x-e^{-x}}{2}\quad,\quad 1+(y')^2 = 1+\frac{e^{2x}-2+e^{-2x}}{4} = \frac{e^{2x}+2+e^{-2x}}{4} = \left(\frac{e^x+e^{-x}}{2}\right)^2$$

$$s = \int_0^2 \sqrt{1+(y')^2}\,dx = \int_0^2\left(\frac{e^x+e^{-x}}{2}\right)dx = \left[\frac{e^x-e^{-x}}{2}\right]_0^2 = \frac{1}{2}\left(e^2-\frac{1}{e^2}\right) \approx 3.627$$

11.

Let $y = \lim_{x\to 0^+}(\sin x)^x$,

$$\ln y = \lim_{x\to 0^+} x\ln(\sin x) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x} = \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-1/x^2} = \lim_{x\to 0^+}\frac{-x^2}{\tan x} = \lim_{x\to 0^+}\frac{-2x}{\sec^2 x} = 0$$

So, $y = \lim_{x\to 0^+}(\sin x)^x = e^0 = 1$

12.

$$\int\frac{2}{\sqrt{x}(x+4)}dx = 4\int\frac{1}{\left(\sqrt{x}\right)^2+2^2}\cdot\frac{1}{2\sqrt{x}}dx = 4\cdot\frac{1}{2}\arctan\frac{\sqrt{x}}{2} = 2\arctan\frac{\sqrt{x}}{2}+C$$

$$\int_0^\infty \frac{2}{\sqrt{x}\,(x+4)}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{2}{\sqrt{x}\,(x+4)}\,dx + \lim_{b\to\infty}\int_1^b \frac{2}{\sqrt{x}\,(x+4)}\,dx$$

$$= \lim_{a\to 0^+}\left[2\arctan\frac{\sqrt{x}}{2}\right]_a^1 + \lim_{b\to\infty}\left[2\arctan\frac{\sqrt{x}}{2}\right]_1^b$$

$$= 2\arctan\frac{1}{2}-0+2\cdot\frac{\pi}{2}-2\arctan\frac{1}{2} = \pi$$

106-2 微積分會考解答

1.

$$I_1=\int_0^1 x^2\,10^{(x^3+1)}\,dx=\frac{1}{3}\int_{x=0}^{x=1}\exp[\ln(10^{(x^3+1)})]d(x^3+1)=\frac{1}{3}\int_{x=0}^{x=1}\exp[(x^3+1)\ln(10)]d(x^3+1)$$

$$=\frac{1}{3\ln 10}\left[10^{(x^3+1)}\right]_{x=0}^{x=1}=\frac{30}{\ln 10}.\quad \#\text{QED}$$

2. $u=\ln(x+1),\ dv=x^2dx,\ du=\frac{1}{x+1}dx,\ v=\frac{x^3}{3}$

$$I_2=\int x^2\ln(x+1)dx=\frac{x^3}{3}\ln(x+1)-\frac{1}{3}\int\frac{x^3}{x+1}dx=\frac{x^3}{3}\ln(x+1)-\frac{1}{3}\int\left(x^2-x+1-\frac{1}{x+1}\right)dx$$

$$=\frac{x^3}{3}\ln(x+1)+\frac{1}{3}\ln(x+1)-\frac{1}{9}x^3+\frac{1}{6}x^2-\frac{1}{3}x+C$$

$$=\frac{1}{3}(x^3+1)\ln(x+1)-\frac{1}{9}x^3+\frac{1}{6}x^2-\frac{1}{3}x+C.\quad \#\text{QED}$$

(Check: $\frac{dI_2}{dx}=\frac{1}{3}\frac{x^3+1}{x+1}+x^2\ln(x+1)-\frac{1}{3}(x^2-x+1)=x^2\ln(x+1)$)

3. $u=\tan^{-1}x^3,\ dv=x^2dx,\ du=\frac{3x^2}{x^6+1}dx,\ v=\frac{x^3}{3}$

$$I_3=\int_0^1 x^2\arctan x^3dx=\left[\frac{1}{3}x^3\tan^{-1}x^3\right]_0^1-\int_0^1\frac{x^5}{x^6+1}dx=\frac{\pi}{12}-\left[\frac{1}{6}\ln(x^6+1)\right]_0^1$$

$$=\frac{\pi}{12}-\frac{\ln 2}{6}.\quad \#\text{QED}$$

4.

$$I_4=\int\tan^5 x\sec^3 xdx=\int\tan^4 x\sec^2 x\,d(\sec x)=\int(\sec^2 x-1)^2\sec^2 x\,d(\sec x)$$

$$=\int(\sec^6 x-2\sec^4 x+\sec^2 x)\,d(\sec x)=\frac{\sec^7 x}{7}-\frac{2\sec^5 x}{5}+\frac{\sec^3 x}{3}+C.\quad \#\text{QED}$$

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$$I_5=\int\frac{dx}{\sqrt{x^2+4x+13}}=\int\frac{d(x+2)}{\sqrt{(x+2)^2+9}}\underset{d(x+2)=3\sec^2\theta d\theta}{\overset{x+2=3\tan\theta}{=}}\int\frac{3\sec^2\theta}{3\sec\theta}d\theta$$

$$=\int\sec\theta d\theta=\ln|\sec\theta+\tan\theta|+K=\ln\left|\frac{\sqrt{x^2+4x+13}}{3}+\frac{x+2}{3}\right|+K$$

$$=\ln\left|\sqrt{x^2+4x+13}+x+2\right|+C.\quad \#\text{QED}$$

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$$I_6 = \int \frac{x^2}{\sqrt{9-x^2}}\,dx \underset{\substack{x=3\sin\theta \\ dx=3\cos\theta d\theta}}{=} \int \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta\, d\theta = \frac{9}{2}\int [1-\cos(2\theta)]\,d\theta$$

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Check:

$$\frac{dI_6}{dx} = \frac{9}{2}\frac{1/3}{\sqrt{1-(x/3)^2}} - \frac{1}{2}\sqrt{9-x^2} + \frac{1}{2}x\frac{x}{\sqrt{9-x^2}} = \frac{9}{2}\frac{1}{\sqrt{9-x^2}} - \frac{1}{2}\sqrt{9-x^2} + \frac{1}{2}\frac{x^2}{\sqrt{9-x^2}}$$

$$= \frac{1}{2}\frac{9-(9-x^2)+x^2}{\sqrt{9-x^2}} = \frac{x^2}{\sqrt{9-x^2}}$$

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$$I_7 = \int \frac{2x^2 - x + 3}{x^3 - 2x^2 + x}\,dx = \int \left[\frac{3}{x} - \frac{1}{x-1} + \frac{4}{(x-1)^2}\right] dx$$

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$$A = \int_0^4 2\sqrt{x}(4-x)\,dx = \int_0^4 (8x^{1/2} - 2x^{3/2})\,dx = \left[\frac{8x^{3/2}}{3/2} - \frac{2x^{5/2}}{5/2}\right]_0^4 = \frac{16}{3}(4)^{3/2} - \frac{4}{5}(4)^{5/2}$$

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(c)

$$V=\int_0^4 \pi\left\{\left[\sqrt{x}(4-x)+4\right]^2-\left[-\sqrt{x}(4-x)+4\right]^2\right\}dx=\pi\int_0^4 16\sqrt{x}(4-x)\,dx$$

$$=16\pi\int_0^4 (4x^{1/2}-x^{3/2})\,dx=16\pi\left[\frac{4x^{3/2}}{3/2}-\frac{x^{5/2}}{5/2}\right]_0^4=16\pi\left[\frac{8}{3}(4)^{3/2}-\frac{2}{5}(4)^{5/2}\right]$$

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$$s=\int_0^2\sqrt{1+(y')^2}\,dx=\int_0^2\left(\frac{e^x+e^{-x}}{2}\right)dx=\left[\frac{e^x-e^{-x}}{2}\right]_0^2=\frac{1}{2}\left(e^2-\frac{1}{e^2}\right)\approx 3.627$$

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So, $y=\lim_{x\to 0^+}(\sin x)^x=e^0=1$

11.

$$\int\frac{2}{\sqrt{x}(x+4)}dx=4\int\frac{1}{\left(\sqrt{x}\right)^2+2^2}\cdot\frac{1}{2\sqrt{x}}dx=4\cdot\frac{1}{2}\arctan\frac{\sqrt{x}}{2}=2\arctan\frac{\sqrt{x}}{2}+C$$

$$\int_0^\infty\frac{2}{\sqrt{x}\,(x+4)}dx=\lim_{a\to 0^+}\int_a^1\frac{2}{\sqrt{x}\,(x+4)}dx+\lim_{b\to\infty}\int_1^b\frac{2}{\sqrt{x}\,(x+4)}dx$$

$$=\lim_{a\to 0^+}\left[2\arctan\frac{\sqrt{x}}{2}\right]_a^1+\lim_{b\to\infty}\left[2\arctan\frac{\sqrt{x}}{2}\right]_1^b$$

$$=2\arctan\frac{1}{2}-0+2\cdot\frac{\pi}{2}-2\arctan\frac{1}{2}=\pi$$