# 國 立 雲 林 科 技 大 學

106 學年度第一學期工學院會考試題

時間：2018/01/10 10:10~12:00
科目：微積分(一)
班級：

❖ 本次會考**禁止使用計算機。**

1. 試找出$f(x)$在以下給定區間內的絕對最大值(absolute maximum)和絕對最小值(absolute minimum)。(本題 10 分)

$$f(x)=x^a(1-x)^b, 0\le x\le 1, a>0, b>0$$

2. $f(x)=1+2x+x^3+4x^5$ ，
試證明: (a) $f(x)$至少有一實根 (5 分)；(b) $f(x)$恰好只有一個實根 (5 分)。

3. $f(x)=x+\cos x, -2\pi\le x\le 2\pi.$ (本大題共 15 分)
試求: (a)遞增及遞減的區間。(4 分)
(b)局部最大值(local maximum)和局部最小值(local minimum)。(4 分)
(c)凹向上(concave upward)與凹向下(concave downward)的區間、及反曲點(inflection point)。(4 分)
(d)繪出 $f(x)$的函數圖形。(3 分)

4. 試求出通過點(3, 5)且在第一象限圍成最小面積的直線的方程式。(本題 10 分)

5. 試計算下列函數之定積分 (本大題共 18 分，每小題 6 分)
(a) $\int_1^2 (8x^3+3x^2)dx$
(b) $\int_{-\pi/4}^{\pi/4} \frac{x^4(\tan x)}{2+\cos x}dx$
(c) $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

6. 一連續函數 $f$ 滿足下式關係，試求解函數$f(x)$ =? (本題 7 分)

$$\int_2^x f(t)dt=(x-1)e^{2x}+\int_3^x e^{-t}f(t)dt \qquad (\text{for all } x)。$$

7. 試計算下列函數之不定積分 (本大題共 30 分，每小題 6 分)
(a) $\int \frac{x}{(1-x^2)^3}dx$ ; (b) $\int \frac{1}{\sqrt{x}(1-3\sqrt{x})}dx$ ; (c) $\int e^{-x}\tan(e^{-x})\,dx$
(d) $\int (1+\sec\pi x)^2(\sec\pi x)(\tan\pi x)dx$
(e) $\int \sin x \sin(\cos x)\,dx$

<<**試題完**>>

1. 試找出$f(x)$在以下給定區間內的絕對最大值(absolute maximum)和絕對最小值(absolute minimum)。(本題 10 分)

$$f(x) = x^a(1-x)^b, 0 \le x \le 1, a > 0, b > 0$$

$f(x) = x^a(1-x)^b, \quad 0 \le x \le 1, a > 0, b > 0.$

$f'(x) = x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1}\,[x \cdot b(-1) + (1-x) \cdot a]$

$= x^{a-1}(1-x)^{b-1}(a - ax - bx)$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of $f$]. In the interval $(0,1)$, $f'(x) = 0 \iff x = \dfrac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

So $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

2. $f(x)=1+2x+x^3+4x^5$ ，

試證明: (a) $f(x)$至少有一實根 (5 分)；(b) $f(x)$恰好只有一個實根 (5 分)。

Let $f(x) = 1 + 2x + x^3 + 4x^5$. Then $f(-1) = -6 < 0$ and $f(0) = 1 > 0$. Since $f$ is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number $c$ between $-1$ and 0 such that $f(c) = 0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots $a$ and $b$ with $a < b$. Then $f(a) = f(b) = 0$. Since $f$ is a polynomial, it is differentiable on $(a, b)$ and continuous on $[a, b]$. By Rolle's Theorem, there is a number $r$ in $(a, b)$ such that $f'(r) = 0$. But $f'(x) = 2 + 3x^2 + 20x^4 \ge 2$ for all $x$, so $f'(x)$ can never be 0. This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

3. $f(x) = x + \cos x, -2\pi \le x \le 2\pi$. (本大題共 15 分)

試求: (a)遞增及遞減的區間。(4 分)

(b)局部最大值(local maximum)和局部最小值(local minimum)。(4 分)

(c)凹向上(concave upward)與凹向下(concave downward)的區間、及反曲點(inflection point)。(4 分)

(d)繪出 $f(x)$的函數圖形。(3 分)

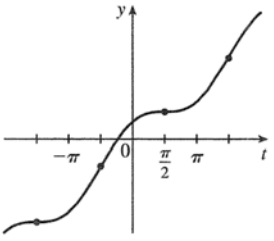
(a) $f(t) = t + \cos t, -2\pi \le t \le 2\pi \Rightarrow f'(t) = 1 - \sin t \ge 0$ for all $t$ and $f'(t) = 0$ when $\sin t = 1 \iff t = -\frac{3\pi}{2}$ or $\frac{\pi}{2}$, so $f$ is increasing on $(-2\pi, 2\pi)$.

(b) No maximum or minimum

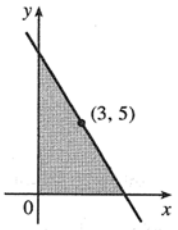
(c) $f''(t) = -\cos t > 0 \iff t \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, so $f$ is CU on these intervals and CD on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$.

There are IPs at $\left(\pm\frac{3\pi}{2}, \pm\frac{3\pi}{2}\right)$ and $\left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$.

(d)



4. 試求出通過點(3, 5)且在第一象限圍成最小面積的直線的方程。(本題 10 分)




The line with slope $m$ (where $m < 0$) through $(3, 5)$ has equation $y - 5 = m(x - 3)$ or $y = mx + (5 - 3m)$. The $y$-intercept is $5 - 3m$ and the $x$-intercept is $-5/m + 3$. So the triangle has area $A(m) = \frac{1}{2}(5 - 3m)(-5/m + 3) = 15 - 25/(2m) - \frac{9}{2}m$. Now

$$A'(m) = \frac{25}{2m^2} - \frac{9}{2} = 0 \quad \Leftrightarrow \quad m^2 = \tfrac{25}{9} \quad \Rightarrow \quad m = -\tfrac{5}{3} \text{ (since } m < 0\text{).}$$

$A''(m) = -\dfrac{25}{m^3} > 0$, so there is an absolute minimum when $m = -\frac{5}{3}$. Thus, an equation of the line is $y - 5 = -\frac{5}{3}(x - 3)$ or $y = -\frac{5}{3}x + 10$.

5(a) $\int_1^2 (8x^3 + 3x^2)dx$ (6 分)

$$\int_1^2 (8x^3 + 3x^2)\,dx = \left[8 \cdot \tfrac{1}{4}x^4 + 3 \cdot \tfrac{1}{3}x^3\right]_1^2 = \left[2x^4 + x^3\right]_1^2 = (2 \cdot 2^4 + 2^3) - (2 + 1) = 40 - 3 = 37$$

5(b) $\int_{-\pi/4}^{\pi/4} \frac{x^4(\tan x)}{2+\cos x}dx$ (6 分)

$\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt = 0$ by Theorem 5.5.7(b), since $\displaystyle f(t) = \frac{t^4 \tan t}{2 + \cos t}$ is an odd function.

5(c) $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$ (6 分)

Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2 - 1) = 2.$$

6. 一連續函數 $f$ 滿足下式關係，試求解函數 $f(x)$ =? (本題 7 分)

$$\int_2^x f(t)dt = (x - 1)e^{2x} + \int_3^x e^{-t}f(t)dt \qquad (\text{for all } x)。$$

Using FTC1, we differentiate both sides of the given equation, $\int_1^x f(t)\,dt = (x - 1)e^{2x} + \int_1^x e^{-t}f(t)\,dt$, and get

$$f(x) = e^{2x} + 2(x - 1)e^{2x} + e^{-x}f(x) \quad \Rightarrow \quad f(x)(1 - e^{-x}) = e^{2x} + 2(x - 1)e^{2x} \quad \Rightarrow \quad f(x) = \frac{e^{2x}(2x - 1)}{1 - e^{-x}}.$$

7(a) $\int \frac{x}{(1-x^2)^3}dx = ?$ (每小題 6 分)

$$\int \frac{x}{(1-x^2)^3}\,dx = -\frac{1}{2}\int (1-x^2)^{-3}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{-2}}{-2} + C = \frac{1}{4(1-x^2)^2} + C$$

**Check:** $\frac{d}{dx}\left[\frac{1}{4(1-x^2)^2} + C\right] = \frac{1}{4}(-2)(1-x^2)^{-3}(-2x) = \frac{x}{(1-x^2)^3}$

7(b) $\int \frac{1}{\sqrt{x}(1-3\sqrt{x})}dx = ?$

$$u = 1 - 3\sqrt{x},\ du = \frac{-3}{2\sqrt{x}}$$

$$\int \frac{1}{\sqrt{x}(1-3\sqrt{x})}\,dx = -\frac{2}{3}\int \frac{1}{1-3\sqrt{x}}\left(\frac{-3}{2\sqrt{x}}\right)dx$$

$$= -\frac{2}{3}\ln\left|1-3\sqrt{x}\right| + C$$

7(c) $\int e^{-x}\tan(e^{-x})\,dx$

$$\int e^{-x}\tan(e^{-x})\,dx = -\int \tan(e^{-x})(-e^{-x})\,dx$$

$$= -\left(-\ln\left|\cos(e^{-x})\right|\right) + C$$

$$= \ln\left|\cos(e^{-x})\right| + C$$

7(d) $\int (1+\sec\pi x)^2(\sec\pi x)(\tan\pi x)dx$

$$\int (1+\sec\pi x)^2 \sec\pi x\tan\pi x\,dx = \frac{1}{\pi}\int (1+\sec\pi x)^2(\pi\sec\pi x\tan\pi x)\,dx = \frac{1}{3\pi}(1+\sec\pi x)^3 + C$$

7(e) $\int \sin x\sin(\cos x)\,dx$

Let $u = \cos x$. Then $du = -\sin x\,dx$ and $-du = \sin x\,dx$, so

$$\int \sin x\sin(\cos x)\,dx = \int \sin u\,(-du) = (-\cos u)(-1) + C = \cos(\cos x) + C.$$