國 立 雲 林 科 技 大 學

113 學年度第二學期工學院會考試題

時間：10:10~12:00

科目：微積分

班級：

※本次測驗禁止使用電子計算機

1. 求下列積分：　(6% × 10 = 60%)

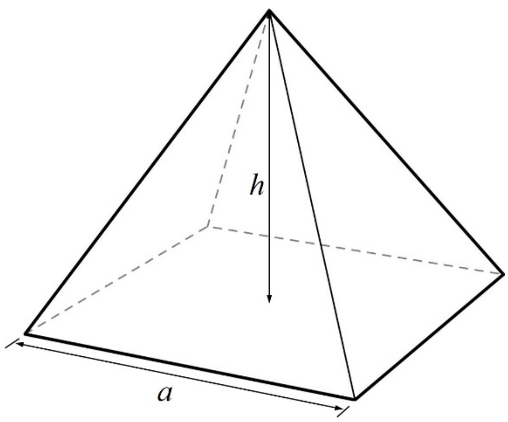
(a) $\int x\cot(x^2+1)dx$　(b) $\int \frac{1}{\sqrt{3+2x-x^2}}dx$　(c) $\int \ln(x^2+1)\,dx$

(d) $\int e^{-x}\sin 2x\,dx$　(e) $\int (1+\sin 2\theta)^2\,d\theta$　(f) $\int \sec^6 2\theta\,d\theta$

(g) $\int \sqrt{4-x^2}\,dx$　(h) $\int \frac{1}{(x^2+4)^{3/2}}dx$　(i) $\int \frac{x^2+x+8}{(x-1)(x^2+4)}dx$

(j) $\int_0^{\infty} \frac{1}{\sqrt{x}\,(1+x)}dx$

2. 一金字塔之底部是邊長為 a 的正方形，高為 h，使用切片積分法證明其體積 $V=\frac{1}{3}a^2h$。　(10%)



3. 分別以 (a) 圓柱殼法 (10%) 及 (b) 圓盤法 (10%)

求以下 3 條線所包圍的區域繞 $x=-2$ 軸旋轉所得旋轉體的體積。

$y=4-x,\quad y=0,\quad x=0$

4. 求以下曲線的弧長。　(10%)

$y=\ln(\cos x),\quad 0\le x\le\frac{\pi}{3}$

《試題完》

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※ The use of scientific calculators is not permitted in examinations.

1. Find the integral. (6% × 10 = 60%)

(a) $\int x\cot(x^2+1)dx$ (b) $\int \frac{1}{\sqrt{3+2x-x^2}}dx$ (c) $\int \ln(x^2+1)\,dx$

(d) $\int e^{-x}\sin 2x\,dx$ (e) $\int (1+\sin 2\theta)^2\,d\theta$ (f) $\int \sec^6 2\theta\,d\theta$

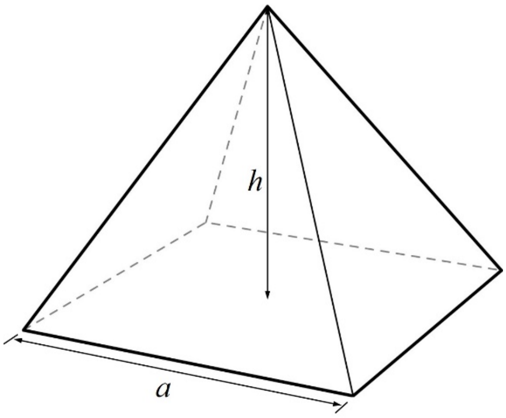
(g) $\int \sqrt{4-x^2}\,dx$ (h) $\int \frac{1}{(x^2+4)^{3/2}}dx$ (i) $\int \frac{x^2+x+8}{(x-1)(x^2+4)}dx$

(j) $\int_0^{\infty} \frac{1}{\sqrt{x}\,(1+x)}dx$

2. Prove that the volume of a pyramid with a square base is

$$V=\frac{1}{3}a^2h$$

where a is the length of the sides of the base and h is the height of the pyramid. (10%)



3. Find the volume of the solid obtained by rotating the region bounded by the graphs of

$$y=4-x,\quad y=0 \quad \text{and} \quad x=0$$

about the line $x = -2$.

(a) Use the shell method. (10%)

(b) Use the disk method. (10%)

4. Find the exact length of the curve. (10%)

$$y=\ln(\cos x),\quad 0\le x\le \frac{\pi}{3}$$

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標準答案試算表

No. 1

1. (a) $\int x\cot(x^2+1)dx = \frac{1}{2}\int \frac{\cos(x^2+1)}{\sin(x^2+1)}\cdot 2x\,dx = \frac{1}{2}\ln\left|\sin(x^2+1)\right| + C$

(b) $\int \frac{1}{\sqrt{3+2x-x^2}}dx = \int \frac{1}{\sqrt{2^2-(x-1)^2}}dx = \arcsin\left(\frac{x-1}{2}\right) + C$

(c) Let $u=\ln(x^2+1)$ and $dv=dx \rightarrow du=\frac{2x}{x^2+1}dx$ and $v=x$

$$\int \ln(x^2+1)\,dx = x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}dx = x\ln(x^2+1) - \int\left(2-\frac{2}{x^2+1}\right)dx$$

$$= x\ln(x^2+1) - 2x + 2\arctan x + C$$

(d) Let $dv=e^{-x}dx$ and $u=\sin 2x \rightarrow v=-e^{-x}$ and $du=2\cos 2x\,dx$

$$\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\,dx \qquad (1)$$

Let $dv=e^{-x}dx$ and $u=\cos 2x \rightarrow v=-e^{-x}$ and $du=-2\sin 2x\,dx$

$$\int e^{-x}\cos 2x\,dx = -e^{-x}\cos 2x - 2\int e^{-x}\sin 2x\,dx \qquad (2)$$

Substitute equation (2) into equation (1).

$$\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x + 2\left(-e^{-x}\cos 2x - 2\int e^{-x}\sin 2x\,dx\right)$$

$$5\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x - 2e^{-x}\cos 2x$$

$$\int e^{-x}\sin 2x\,dx = -\frac{1}{5}e^{-x}\sin 2x - \frac{2}{5}e^{-x}\cos 2x + C$$

(e) $\int (1+\sin 2\theta)^2\,d\theta = \int (1+2\sin 2\theta+\sin^2 2\theta)\,d\theta$

$$= \int\left(1+2\sin 2\theta+\frac{1-\cos 4\theta}{2}\right)d\theta = \frac{3}{2}\theta - \cos 2\theta - \frac{\sin 4\theta}{8} + C$$

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標準答案試算表

No. 2

(f) $\int \sec^6 2\theta\, d\theta = \int (\sec^2 2\theta)^2 \cdot \sec^2 2\theta\, d\theta = \int \frac{(1+\tan^2 2\theta)^2}{2} \cdot 2\sec^2 2\theta\, d\theta$

$$= \int \left(\frac{1}{2} + \tan^2 2\theta + \frac{1}{2}\tan^4 2\theta\right) 2\sec^2 2\theta\, d\theta = \frac{\tan 2\theta}{2} + \frac{\tan^3 2\theta}{3} + \frac{\tan^5 2\theta}{10} + C$$

(g) Let $x = 2\sin\theta$. Then, $dx = 2\cos\theta\, d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$

$$\int \sqrt{4-x^2}\, dx = \int 2\cos\theta \cdot 2\cos\theta\, d\theta = \int 4\cos^2\theta\, d\theta = \int 4 \cdot \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int (2 + 2\cos 2\theta) d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\arcsin\frac{x}{2} + 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} + C = 2\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C$$

(h) Let $x = 2\tan\theta$. Then, $dx = 2\sec^2\theta\, d\theta$ and $\sqrt{x^2+4} = 2\sec\theta$

$$\int \frac{1}{(x^2+4)^{3/2}} dx = \int \frac{1}{(2\sec\theta)^3}\, 2\sec^2\theta\, d\theta = \int \frac{1}{4\sec\theta} d\theta = \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta + C = \frac{1}{4} \cdot \frac{x}{\sqrt{x^2+4}} + C$$

(i) $\frac{x^2+x+8}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{2}{x-1} + \frac{-x}{x^2+4}$

$$\int \frac{x^2+x+8}{(x-1)(x^2+4)} dx = \int \left(\frac{2}{x-1} - \frac{x}{x^2+4}\right) dx = 2\ln|x-1| - \frac{1}{2}\ln(x^2+4) + C$$

(j) $\int \frac{1}{\sqrt{x}(1+x)} dx = 2\int \frac{1}{1+\left(\sqrt{x}\right)^2} \cdot \frac{1}{2\sqrt{x}} dx = 2\arctan\sqrt{x} + C$

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)} dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}(1+x)} dx + \lim_{b\to\infty} \int_1^\infty \frac{1}{\sqrt{x}(1+x)} dx$$

$$= \lim_{a\to 0^+} \left[2\arctan\sqrt{x}\right]_a^1 + \lim_{b\to\infty} \left[2\arctan\sqrt{x}\right]_1^b = 2\cdot\frac{\pi}{4} - 0 + 2\cdot\frac{\pi}{2} - 2\cdot\frac{\pi}{4} = \pi$$

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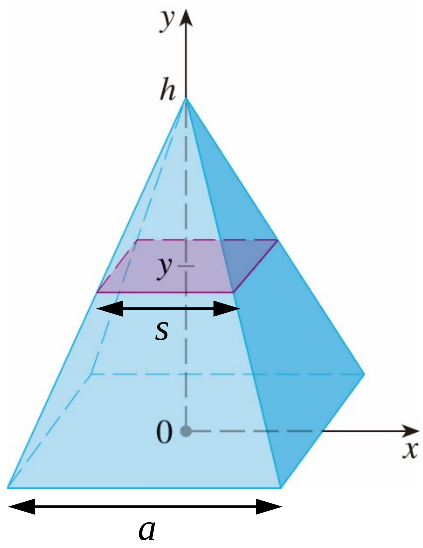
標準答案試算表

No. 3

2. 一金字塔之底部是邊長為 a 的正方形，高為 h，使用積分法證明其體積 $V=\frac{1}{3}a^2h$。 (10%)



$$\frac{h-y}{s}=\frac{h}{a}$$

$$V=\int_0^h s^2\,dy=\int_0^h\left[\frac{a(h-y)}{h}\right]^2 dy=\int_0^h\frac{a^2}{h^2}(h-y)^2\,dy$$

$$=\frac{a^2}{h^2}\int_0^h(h^2-2hy+y^2)\,dy=\frac{a^2}{h^2}\left[h^2y-hy^2+\frac{1}{3}y^3\right]_0^h$$

$$=\frac{a^2h}{3}$$

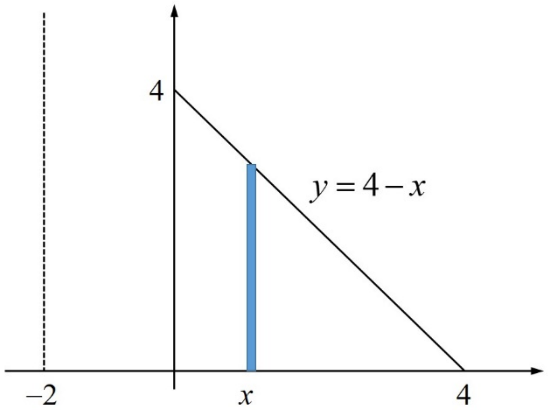
3. 分別以 (a) 圓柱殼法 (10%) 及 (b) 圓盤法 (10%)

求以下 3 條線所包圍的區域繞 $x=-2$ 軸旋轉所得旋轉體的體積。

$y=4-x$， $y=0$， $x=0$

(a) 圓柱殼法

半徑：$x-(-2)=x+2$， 高：$4-x$



$$V=\int_0^4 2\pi(x+2)(4-x)dx$$

$$=2\pi\int_0^4(-x^2+2x+8)dx$$

$$=2\pi\left[-\frac{x^3}{3}+x^2+8x\right]_0^4$$

$$=2\pi\left(-\frac{64}{3}+16+32\right)=\frac{160}{3}\pi$$

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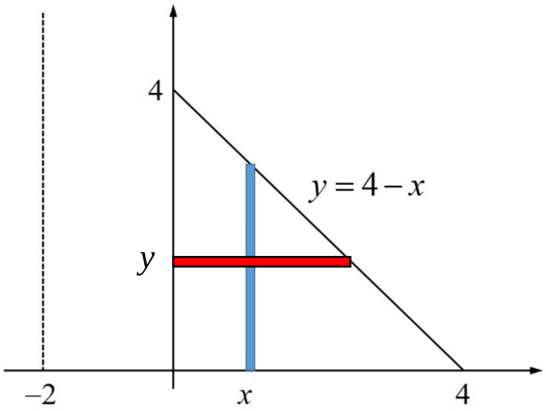
標準答案試算表

No. 4

(b) 圓盤法

外徑：$R(y) = 4 - y - (-2) = 6 - y$

內徑：$r(y) = 0 - (-2) = 2$



$$V = \int_0^4 \pi\left[(6-y)^2 - 2^2\right]dy$$

$$= \pi\int_0^4 (32 - 12y + y^2)dy$$

$$= \pi\left[32y - 6y^2 + \frac{y^3}{3}\right]_0^4$$

$$= \pi\left(128 - 96 + \frac{64}{3}\right) = \frac{160}{3}\pi$$

4. 求以下曲線的弧長。 (10%)

$$y = \ln(\cos x), \quad 0 \le x \le \frac{\pi}{3}$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$\int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx$$

$$= \int_0^{\pi/3} \sec x\,dx = \left[\ln\left|\sec x + \tan x\right|\right]_0^{\pi/3}$$

$$= \ln(2 + \sqrt{3}) - \ln(1 + 0) = \ln(2 + \sqrt{3})$$